Chapter 7

Viscous Flow

This chapter will focus on problems in which viscous stress plays an important role in determining the motion of the fluid. The topic in general is quite broad; to gain understanding of the fundamental physics, we will restrict our attention to the following limits:

- incompressible fluid
- isotropic Newtonian fluid with constant properties
- at most two-dimensional unsteady flow

The chapter will consider the governing equations and then solve a few representative problems.

7.1 Governing Equations

This section considers the governing equations for the conditions specified for this chapter. In dimensional non-conservative form, the governing equations are as follows:

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &= -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} &= -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \\
\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) &= \left(\frac{\partial P}{\partial t} + u \frac{\partial P}{\partial x} + v \frac{\partial P}{\partial y} \right) + k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \\
&\quad + 2\mu \left(\left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right)
\end{aligned}
$$

An argument could be made to eliminate the viscous dissipation term and the pressure derivatives in the energy equation. The argument is subtle and based on the low Mach number limit which corresponds to incompressibility.

7.2 Couette Flow

Consider a channel flow driven by plate motion See Figure 7.1

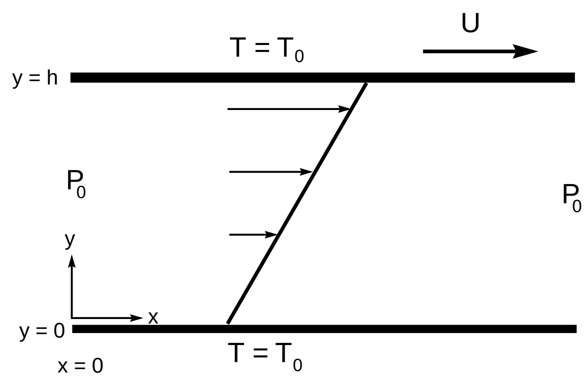

Figure 7.1: Sketch for Couette flow

The mechanics of such a flow can be described by stripping away many extraneous terms from the governing equations.

Take

- fully developed velocity and temperature profiles: $\frac{\partial u}{\partial x} \equiv 0$, $\frac{\partial T}{\partial x} \equiv 0$
- steady flow $\frac{\partial}{\partial t} \equiv 0$
- constant pressure field $P(x, y, t) = P_o$
- constant temperature channel walls $T(x, 0, t) = T(x, h, t) = T_o$

Since fully developed mass gives:

$$\frac{\partial v}{\partial y} = 0 \tag{7.1}$$

$$v(x, y) = f(x) \tag{7.2}$$

and since in order to prevent mass flowing through the wall boundaries, $v(x, 0) = v(x, h) = 0$, thus

$$v(x,y) = 0 \tag{7.3}$$

Since $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial t} = 0$ and $\frac{\partial T}{\partial x} = 0, \frac{\partial T}{\partial t} = 0$, we have at most,

$$u = u(y) \tag{7.4}$$
$$T = T(y) \tag{7.5}$$

The y momentum equation has no information and x momentum and energy reduce to the following:

$$0 = \mu \frac{d^2u}{dy^2} \tag{7.6}$$
$$0 = k\frac{d^2T}{dy^2} + \mu \left(\frac{du}{dy}\right)^2 \tag{7.7}$$

The x momentum equation is thus

$$\frac{d^2u}{dy^2} = 0 \tag{7.8}$$
$$\frac{du}{dy} = C_1 \tag{7.9}$$
$$u(y) = C_1 y + C_2 \tag{7.10}$$

Now applying $u(0) = 0$ and $u(h) = U$ to fix C_1 and C_2 we get

$$u(y) = U\frac{y}{h} \tag{7.11}$$

Shear stress:

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \tag{7.12}$$
$$\tau_{yx} = \mu \frac{U}{h} \tag{7.13}$$

The energy equation becomes

$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\left(\frac{du}{dy}\right)^2 \tag{7.14}$$

$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\frac{U^2}{h^2} \tag{7.15}$$

$$\frac{dT}{dy} = -\frac{\mu}{k}\frac{U^2}{h^2}\ y + C_1 \tag{7.16}$$

$$T(y) = -\frac{1}{2}\frac{\mu}{k}\frac{U^2}{h^2}\ y^2 + C_1 y + C_2 \tag{7.17}$$

Now $T(0) = T_o$ and $T(h) = T_o$. This fixes the constants, so

$$T(y) = \frac{1}{2}\frac{\mu}{k}U^2\left(\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right) + T_o \tag{7.18}$$

In dimensionless form this becomes

$$\frac{T - T_o}{T_o} = \frac{1}{2}\frac{\mu c_p}{k}\frac{U^2}{c_p T_o}\left(\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right) \tag{7.19}$$

$$\frac{T - T_o}{T_o} = \frac{Pr\ Ec}{2}\left(\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right) \tag{7.20}$$

$$\text{Prandtl Number:} \qquad Pr \equiv \frac{\mu c_p}{k} = \frac{\frac{\mu}{\rho}}{\frac{k}{\rho c_p}} = \frac{\nu}{\alpha} = \frac{momentum\ diffusivity}{thermal\ diffusivity} \tag{7.21}$$

$$\text{Eckert Number:} \qquad Ec \equiv \frac{U^2}{c_p T_o} = \frac{kinetic\ energy}{thermal\ energy} \tag{7.22}$$

Now

$$\frac{dT}{dy} = \frac{1}{2}\frac{\mu}{k}\frac{U^2}{h^2}(h - 2y) \tag{7.23}$$

$$q_y = -k\frac{dT}{dy} = \frac{1}{2}\mu\frac{U^2}{h^2}(2y - h) \tag{7.24}$$

$$q_y(0) = -\frac{\mu U^2}{2h} \tag{7.25}$$

Note:

- at lower wall, heat flux into wall; heat generated in fluid conducted to wall
- wall heat flux magnitude independent of thermal conductivity
- higher plate velocity, higher wall heat flux
- higher viscosity, higher wall heat flux
- thinner gap, higher wall heat flux

Also since T_{max} occurs at $y = \frac{h}{2}$

$$T_{max} = \frac{1}{8}\frac{\mu}{k}U^2 + T_o \tag{7.26}$$

Note:

- high viscosity, high maximum temperature
- high plate velocity, high maximum temperature
- low thermal conductivity, high maximum temperature

Dimensionless wall heat flux given by the Nusselt number:

$$Nu \equiv \left| \frac{q_y(0)}{\frac{k\Delta T}{\Delta y}} \right| = \left| \frac{q_y(0)\Delta y}{k\Delta T} \right| \tag{7.27}$$

$$Nu = \frac{\frac{\mu U^2}{2h}\frac{h}{2}}{k\frac{1}{8}\frac{\mu}{k}U^2} = 2 \tag{7.28}$$

7.3 Suddenly Accelerated Flat Plate

The problem of pulling a plate suddenly in a fluid which is initially at rest is often known as Stokes' First Problem or Rayleigh's problem.

7.3.1 Formulation

Consider a channel flow driven by a suddenly accelerated plate. See Figure 7.2 Initially, $t < 0$

- fluid at rest
- plate at rest

For $t \geq 0$

- plate pulled at constant velocity U

Assume:

- constant pressure $P(x, y, t) = P_o$
- fully developed flow $\frac{\partial u}{\partial x} = 0$, $\frac{\partial T}{\partial x} = 0$

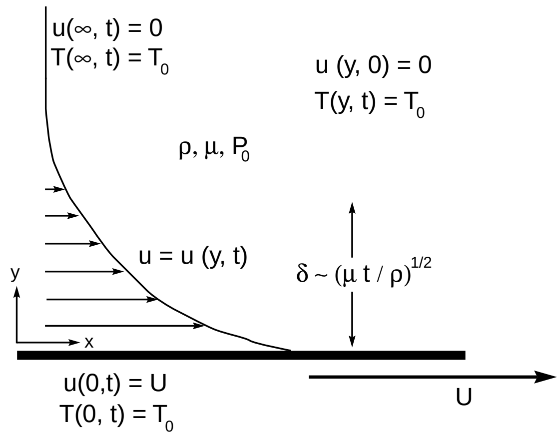

Figure 7.2: Sketch for Stokes' First Problem

Again from mass we deduce that $v(x, y, t) = 0$. The x momentum equation reduces to

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} \tag{7.29}$$

The initial and boundary conditions are

$$u(y, 0) = 0 \tag{7.30}$$
$$u(0, t) = U \tag{7.31}$$
$$u(\infty, t) = 0 \tag{7.32}$$

7.3.2 Velocity Profile

This problem is solved in detail in lecture. The solution for the velocity field is shown to be

$$\frac{u}{U_o} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{y/\sqrt{\nu t}} \exp\left(-s^2\right)\, ds \tag{7.33}$$

7.4 Starting Transient for Plane Couette Flow

The starting transient problem for plane Couette flow can be formulated as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{7.34}$$
$$u(y, 0) = 0, \qquad u(0, t) = U_o, \qquad u(h, t) = 0. \tag{7.35}$$

In class a detailed solution is presented via the technique of separation of variables. The solution is

$$\frac{u}{U_o} = 1 - \frac{y}{h} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(\frac{-n^2\pi^2\nu t}{h^2}\right)\sin\left(\frac{n\pi y}{h}\right) \tag{7.36}$$

7.5 Blasius Boundary Layer

The problem of flow over a flat plate in the absence of pressure gradient is formulated and solved using the classical approach of Blasius.

7.5.1 Formulation

After suitable scaling and definition of similarity variables, discussed in detail in class, the following third order non-linear ordinary differential equation is obtained:

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0, \tag{7.37}$$

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = 0, \qquad \left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 1, \qquad f|_{\eta=0} = 0. \tag{7.38}$$

This equation is solved numerically as a homework problem.

7.5.2 Wall Shear Stress

The solution is used to obtain the classical formulae for skin friction coefficient:

$$C_f = \frac{0.664}{\sqrt{Re_x}}, \tag{7.39}$$

and drag coefficient:

$$C_D = \frac{1.328}{\sqrt{Re_L}}. \tag{7.40}$$