UNIT-IV

Recurrence Relation

Generating Functions:

In mathematics, a **generating function** is a formal power series in one indeterminate, whose coefficients encode information about a sequence of numbers a_n that is indexed by the natural numbers. Generating functions were first introduced by Abraham de Moivre in 1730, in order to solve the general linear recurrence problem. One can generalize to formal power series in more than one indeterminate, to encode information about arrays of numbers indexed by several natural numbers.

Generating functions are not functions in the formal sense of a mapping from a domain to a codomain; the name is merely traditional, and they are sometimes more correctly called **generating series**.

Ordinary generating function

The *ordinary generating function* of a sequence a_n is

$$G(a_n;x)=\sum_{n=0}^{\infty}a_nx^n.$$

When the term *generating function* is used without qualification, it is usually taken to mean an ordinary generating function.

If a_n is the probability mass function of a discrete random variable, then its ordinary generating function is called a probability-generating function.

The ordinary generating function can be generalized to arrays with multiple indices. For example, the ordinary generating function of a two-dimensional array $a_{m,\,n}$ (where n and m are natural numbers) is

$$G(a_{m,n};x,y)=\sum_{m,n=0}^{\infty}a_{m,n}x^my^n.$$

Example:

$$G(n^2;x) = \sum_{n=0}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$$

Exponential generating function

The *exponential generating function* of a sequence a_n is

$$\mathrm{EG}(a_n;x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Example:

$$\mathrm{EG}(n^2;x) = \sum_{n=0}^{\infty} \frac{n^2 x^n}{n!} = x(x+1)e^x$$

Function of Sequences:

Generating functions giving the first few powers of the nonnegative integers are given in the following table.

n^p	$f(x)$	series
1	$\frac{x}{1-x}$	$x + x^2 + x^3 + \ldots$
n	$\frac{x}{(1-x)^2}$	$x + 2x^2 + 3x^3 + 4x^4 + \ldots$
n^2	$\frac{x(x+1)}{(1-x)^3}$	$x + 4x^2 + 9x^3 + 16x^4 + \ldots$
n^3	$\frac{x(x^2+4x+1)}{(1-x)^4}$	$x + 8x^2 + 27x^3 + \ldots$
n^4	$\frac{x(x+1)(x^2+10x+1)}{(1-x)^5}$	$x + 16x^2 + 81x^3 + \ldots$

There are many beautiful generating functions for special functions in number theory. A few particularly nice examples are

$$f(x) = \frac{1}{(x)_\infty} \quad (2)$$

$$= \sum_{n=0}^{\infty} P(n) x^n \quad (3)$$

$$= 1 + x + 2x^2 + 3x^3 + \ldots \quad (4)$$

for the partition function P, where $(q)_\infty$ is a *q*-Pochhammer symbol, and

$$f(x) = \frac{x}{1-x-x^2} \quad (5)$$

$$= \sum_{n=0}^{\infty} F_n x^n \quad (6)$$

$$= x + x^2 + 2x^3 + 3x^4 + \ldots \quad (7)$$

for the Fibonacci numbers F_n.

Generating functions are very useful in combinatorial enumeration problems. For example, the subset sum problem, which asks the number of ways $c_{m,s}$ to select m out of M given integers such that their sum equals s, can be solved using generating functions.

Calculating Coefficient of generating function:

By using the following polynomial expansions, we can calculate the coefficient of a generating function.

Polynomial Expansions:

1) $\frac{1-x^{m+1}}{1-x} = 1 + x + x^2 + \ldots + x^m$

2) $\frac{1}{1-x} = 1 + x + x^2 + \ldots$

3) $(1+x)^n = 1 + C(n,1)x + C(n,2)x^2 + \ldots + C(n,r)x^r + \ldots + C(n,n)x^n$

4) $(1-x^m)^n = 1 - C(n,1)x^m + C(n,2)x^{2m} + \ldots + (-1)^k C(n,k)x^{km} + \ldots + (-1)^n C(n,n)x^{nm}$

5) $\frac{1}{(1-x)^n} = 1 + C(1+n-1,1)x + C(2+n-1,2)x^2 + \ldots + C(r+n-1,r)x^r + \ldots$

6) If h(x)=f(x)g(x), where f(x) $= a_0 + a_1x + a_2x^2 + \ldots$ and g(x) $= b_0 + b_1x + b_2x^2 + \ldots$, then

h(x) $= a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \ldots + (a_rb_0 + a_{r-1}b_1 + a_{r-2}b_2 + \ldots + a_0b_r)x^r + \ldots$

Example

Find the coefficient of x^{16} *in* $(x^2 + x^3 + x^4 + \ldots)^5$

$$x^{16} \textit{ in } x^{10}(1-x)^{-5} [\textit{i.e, the } x^6 \textit{ term in} (1-x)^{-5} \textit{ is}$$

$$\textit{multiplied by } x^{10} \textit{ to become the } x^{16} \textit{ term in } x^{10}(1-x)^{-5}]$$

To simplify the expression, we extract x^2 from each polynomial factor and the apply identity (2).

$$(x^2 + x^3 + x^4 + \ldots)^5 = [x^2(1 + x + x^2 + \ldots)]^5$$
$$= x^{10}(1 + x + x^2 + \ldots)^5$$
$$= x^{10} \frac{1}{(1-x)^5}$$

Thus the coefficient of x^{16} *in* $(x^2 + x^3 + x^4 + \ldots)^5$ is the coefficient of x^{16} in x^{10} (1-x) $^{-5}$ [i.e., the x^6 term in (1-x) $^{-5}$ is multiplied by to become the x^{16} term in x^{10} (1-x) $^{-5}$]

$$\frac{1}{(1-x)^n} = 1 + C(1+n-1,1)x + C(2+n-1,2)x^2 + \ldots + C(r+n-1,r)x^r + \ldots$$

From expansion (5) we see that the coefficient of $x^6 \textit{ in} (1-x)^{-5} \textit{ is } C(6+5-1,6)$

More generally, the coefficient of x^r in

$$x^r \text{ in } x^{10}(1-x)^{-5} \text{ equals the coefficient of } x^{r-10} \text{ in}$$

$$(1-x)^{-5}, \text{namely}, C((r-10)+5-1,(r-10)).$$

Recurrence relations:

Introduction :A recurrence relation is a formula that relates for any integer n ≥ 1, the n-th term of a sequence A = {ar}∞r=0 to one or more of the terms a0,a1,….,an-1. Example. If Sn denotes the sum of the first n positive integers, then

10. Sn = n + Sn-1. Similarly if d is a real number, then the *n*th term of an arithmetic progression with common difference d satisfies the relation

11. an = an -1 + d. Likewise if pn denotes the nth term of a geometric progression with common ratio r, then

[redacted]pn = r pn – 1. We list other examples
as: an – 3an-1 + 2an-2 = 0.
an – 3 an-1+ 2 an-2 = n2 + 1.
an – (n - 1) an-1 - (n - 1) an-2 = 0.
an – 9 an-1+ 26 an-2 – 24 an-3 = 5n.
an – 3(an-1)2 + 2 an-2 = n.
an = a0 an-1+ a1 an-2+ … + an-
1a0. a2n + (an-1)2 = -1.

Definition. Suppose n and k are nonnegative integers. A recurrence relation of the form c0(n)an + c1(n)an-1 + …. + ck(n)an-k = f(n) for n $\geq$ k, where c0(n), c1(n),…., ck(n), and f(n) are functions of n is said to be a **linear recurrence relation**. If c0(n) and ck(n) are not identically zero, then it is said to be a **linear recurrence relation *degree* k**. If c0(n), c1(n),…., ck(n) are constants, then the recurrence relation is known as a **linear relation with constant coefficients**. If f(n) is identically zero, then the recurrence relation is said to be **homogeneous**; otherwise, it is **inhomogeneous.**

Thus, all the examples above are linear recurrence relations except (8), (9), and (10); the relation (8), for instance, is not linear because of the squared term.
The relations in (3), (4) , (5), and (7) are linear with constant coefficients.

Relations (1), (2), and (3) have degree 1; (4), (5), and (6) have degree 2; (7) has degree 3. Relations (3) , (4), and (6) are homogeneous.

There are no general techniques that will enable one to solve all recurrence relations. There are, nevertheless, techniques that will enable us to solve linear recurrence relations with constant coefficients.

SOLVING RECURRENCE RELATIONS BY SUSTITUTION AND GENERATING FUNCTIONS

We shall consider four methods of solving recurrence relations in this and the next two sections:

5. Substitution (also called iteration),
6. Generating functions,
7. Characteristics roots, and
8. Undetermined coefficients.

In the substitution method the recurrence relation for an is used repeatedly to solve for a general expression for an in terms of n. We desire that this expression involve no other terms of the sequence except those given by boundary conditions.

The mechanics of this method are best described in terms of examples. We used this method in Example5.3.4. Let us also illustrate the method in the following examples.

Example

Solve the recurrence relation an = a n-1 + f(n) for n ³1 by substitution

a1= a0 + f(1)
a2 = a1 + f(2) = a0 + f(1) + f(2))

a3 = a2 + f(3)= a0 + f(1) + f(2) + f(3)

.
.
.
an = a0 + f(1) + f(2) +….+
f(n) n
= a0 + ∑ f(k)
K = 1

Thus, an is just the sum of the f(k) „s plus a0.

More generally, if c is a constant then we can solve an = c a n-1 + f(n) for n ³1 in the same way:

a1 = c a0 + f(1)

a2 = c a1 + f(2) = c (c a0 + f(1)) +
f(2) = c2 a0 + c f(1) + f(2)

a3= c a2 + f(3) = c(c 2 a0 + c f(1) + f(2)) +
f(3) =c3 a0 + c2 f(1) + c f(2) + f(3)

.

.

.

an = c a n-1 + f(n) = c(c n-1 a0 + c n-2 f(1) +. . . + c n-2 + f(n-1)) +
f(n) =c n a0 + c n-1 f(1) + c n-2 f(2) +. . .+ c f(n-1) + f(n)

Or

an = c n a0 + ∑c n-k f(k)

Solution of Linear Inhomogeneous Recurrence Relations:

The equation $an + c1\,an-1 + c2\,an-2 = (n)$, where $c1$and $c2$ are constant, and $f(n)$ is not identically 0, is called a second-order linear **inhomogeneous** recurrence relation (or difference equation) with constant coefficients. The homogeneous case, which we"ve looked at already, occurs when

$(n) \equiv 0$. The inhomogeneous case occurs more frequently. The homogeneous case is so important largely because it gives us the key to solving the inhomogeneous equation. If you"ve studied linear differential equations with constant coefficients, you"ll see the parallel. We will call the difference obtained by setting the right-hand side equal to 0, the "associated homogeneous equation." We know how to solve this. Say that V is a solution. Now suppose that (n) is any particular solution of the inhomogeneous equation. (That is, it solves the equation, but does not necessarily match the initial data.) Then $U=V+(n)$ is a solution to the inhomogeneous equation, which you can see simply by substituting U into the equation. On the other hand, every solution U of the inhomogeneous equation is of the form $U=V+(n)$ where V is a solution of the homogeneous equation, and $g(n)$ is a particular solution of the inhomogeneous equation. The proof of this is straightforward. If we have two solutions to the inhomogeneous equation, say U_1 and U_2, then their difference $U_1-U_2=V$ is a solution to the homogeneous equation, which you can check by substitution. But then $U_1= V + U_2$, and we can set $U_2=(n)$, since by assumption, U_2 is a particular solution. This leads to the following theorem: **the general solution to the inhomogeneous equation is the general solution to the associated homogeneous equation, plus any particular solution to the inhomogeneous equation.** This gives the following procedure for solving the inhomogeneous equation:

4. Solve the associated homogeneous equation by the method we‟ve learned. This will involve variable (or undetermined) coefficients.

5. Guess a particular solution to the inhomogeneous equation. It is because of the guess that I‟ve called this a procedure, not an algorithm. For simple right-hand sides , we can say how to compute a particular solution, and in these cases, the procedure merits the name "algorithm."

6. The general solution to the inhomogeneous equation is the sum of the answers from the two steps above.

7. Use the initial data to solve for the undetermined coefficients from step 1.

To solve the equation $a_n - 6a_{n-1} + 8a_{n-2} = 3$. Let‟s suppose that we are also given the initial data $a_0 = 3$, $a_1 = 3$. The associated homogeneous equation is $a_n - 6a_{n-1} + 8a_{n-2} = 0$, so the characteristic equation is $r^2 - 6r + 8 = 0$, which has roots $r_1 = 2$ and $r_2 = 4$. Thus, the general solution to the associated homogeneous equation is $c_1 2^n + c_2 4$. When the right-hand side is a polynomial, as in this case, there will always be a particular solution that is a polynomial.

Usually, a polynomial of the same degree will work, so we‟ll guess in this case that there is a constant C that solves the homogeneous equation. If that is so, then $a_n = a_{n-1} = a_{n-2} = C$, and substituting into the equation gives $C - 6C + 8C = 3$, and we find that $C = 1$. Now, the general solution to the inhomogeneous equations is $c_1 2^n + c_2 4^n + 1$. Reassuringly, this is the answer given in the back of the book. Our initial data lead to the equations $c_1 + c_2 + 1 = 3$ and $2c_1 + 4c_2 + 1 = 3$, whose solution is $c_1 = 3$, $c_2 = -1$. Finally, the solution to the inhomogeneous equation, with the initial condition given, is $a_n = 3 \cdot 2^n - 4^n + 1$. Sometimes, a polynomial of the same degree as the right-hand side doesn‟t work. This happens when the characteristic equation has 1 as a root. If our equation had been $a_n - 6a_{n-1} + 5a_{n-2} = 3$, when we guessed that the particular solution was a constant C, we‟d have arrived at the equation $C - 6C + 5C = 3$, or $0 = 3$. The way to deal with this is to increase the degree of the polynomial. Instead of assuming that the solution is constant, we‟ll assume that it‟s linear. In fact, we‟ll guess that it is of the form

$gn = nC$. Then we have $nC - 6\ n-1\ C + 5\ n-2\ C = 3$, which simplifies to $6C - 10C = 3$ so that $C = -34$. Thus, $gn = -3\ n4$. This won‟t be enough if 1 is a root of multiplicity 2, that is, if $r - 1\ 2$ is a factor of the characteristic polynomial. Then there is a particular solution of the form $gn = Cn2$. For second-order equations, you never have to go past this. If the right-hand side is a polynomial of degree greater than 0, then the process works juts the same, except that you start with a polynomial of the same degree, increase the degree by 1, if necessary, and then once more, if need be. For example, if the right-hand side were $fn = 2\ n - 1$, we would start by guessing a particular solution $gn = C1\ n + C2$. If it turned out that 1 was a characteristic root, we would amend our guess to $gn = C1\ n2 + C2\ n + C3$. If 1 is a double root, this will fail also, but $gn = C1\ n3 + C2\ n2 + C3\ n + C4$ will work in this case.

Another case where there is a simple way of guessing a particular solution is when the right-hand side is an exponential, say $fn = Cn$. In that case, we guess that a particular solution is just a constant multiple of f, say $(n) = kCn$. Again, we gave trouble when 1 is a characteristic root. We then guess that $gn = knCn$, which will fail only if 1 is a double root. In that case we must use $gn = kn2\ Cn$, which is as far as we ever have to go in the second-order case. These same ideas extend to higher-order recurrence relations, but we usually solve them numerically, rather than exactly. A third-order linear difference equation with constant coefficients leads to a cubic characteristic polynomial. There is a formula for the roots of a cubic, but it‟s very complicated.

For fourth-degree polynomials, there‟s also a formula, but it‟s even worse. For fifth and higher degrees, no such formula exists. Even for the third-order case, the exact solution of a simple-looking inhomogeneous linear recurrence relation with constant coefficients can take pages to write down. The coefficients will be complicated expressions involving square roots and cube roots. For most, if not all, purposes, a simpler answer with numerical coefficients is better, even though they must in the nature of things, be approximate.

The procedure I‟ve suggested may strike you as silly. After all, we‟ve already solved the characteristic equation, so we know whether 1 is a characteristic root, and what it‟s multiplicity is. Why not start with a polynomial of the correct degree? This is all well and good, while you‟re taking the course, and remember the procedure in detail. However, if you have to use this procedure some years from now, you probably won‟t remember all the details. Then the method I‟ve suggested will be valuable. Alternatively, you can start with a general polynomial of the maximum possible degree This leads to a lot of extra work if you‟re solving by hand, but it‟s the approach I prefer for computer solution.