CHAPTER 5

The Brownian motion

The Brownian motion is the most fundamental continuous time stochastic process. It is both a martingale of the type considered in Section 4.2 and a Gaussian process as considered in Section 3.2. It also has continuous sample path, independent increments, and the strong Markov property of Section 6.1. Having all these beautiful properties allows for a rich mathematical theory. For example, many probabilistic computations involving the Brownian motion can be made explicit by solving partial differential equations. Further, the Brownian motion is the corner stone of diffusion theory and of stochastic integration. As such it is the most fundamental object in applications to and modeling of natural and man-made phenomena.

In this chapter we define and construct the Brownian motion (in Section 5.1), then deal with a few of the many interesting properties it has. Specifically, in Section 5.2 we use stopping time and martingale theory to study the hitting times and the running maxima of this process, whereas in Section 5.3 we consider the smoothness and variation of its sample path.

5.1. Brownian motion: definition and construction

Our starting point is an axiomatic definition of the Brownian motion via its Gaussian property.

Definition 5.1.1. *A stochastic process* $(W_t,\ 0 \le t \le T)$ *is called a* Brownian motion *(or a Wiener Process) if:*
(a) W_t *is a Gaussian process*
(b) $\mathbf{E}(W_t) = 0$, $\mathbf{E}(W_t W_s) = \min(t, s)$
(c) For almost every ω*, the sample path,* $t \mapsto W_t(\omega)$ *is continuous on* $[0, T]$.

Note that (a) and (b) of Definition 5.1.1 completely characterize the finite dimensional distributions of the Brownian motion (recall Corollary 3.2.18 that Gaussian processes are characterized by their mean and auto-covariance functions). Adding property (c) to Definition 5.1.1 allows us to characterize its sample path as well. We shall further study the Brownian sample path in Sections 5.2 and 5.3. We next establish the independence of the zero-mean Brownian increments, implying that the Brownian motion is an example of the martingale processes of Section 4.2 (see Proposition 4.2.3). Note however that the Brownian motion is a non-stationary process (see Proposition 3.2.25), though it does have stationary increments.

Proposition 5.1.2. *The Brownian motion has independent increments of zero-mean.*

Proof. From part (b) of Definition 5.1.1, we obtain that for $t \ge s$ and $h > 0$,

$$\mathrm{Cov}(W_{t+h} - W_t, W_s) = \mathbf{E}[(W_{t+h} - W_t)W_s] = \mathbf{E}(W_{t+h}W_s) - E(W_t W_s) = s - s = 0\,.$$

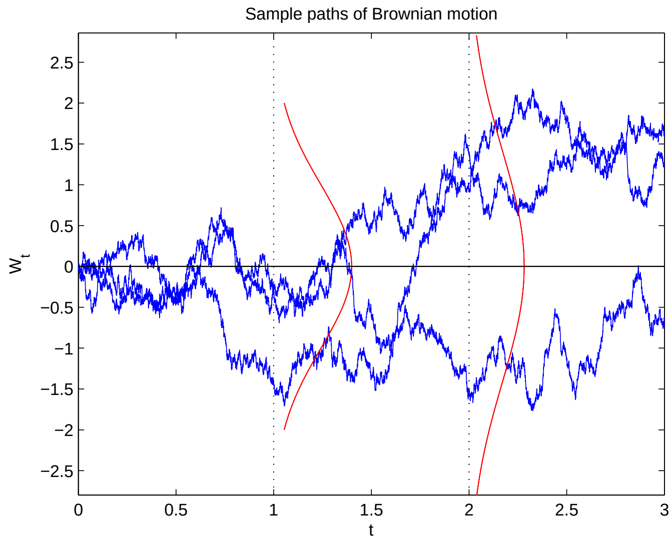

FIGURE 1. Three sample paths of Brownian motion. The density curves illustrate that the random variable W_1 has a $N(0, 1)$ distribution, while W_2 has a $N(0, 2)$ distribution.

Thus, $W_{t+h} - W_t$ and W_s are uncorrelated for any fixed $h > 0$ and $s \leq t$. Fixing $n < \infty$ and $0 \leq s_1 < s_2 < \ldots < s_n \leq t$, since $\{W_t\}$ is a Gaussian process, we know that $(W_{t+h}, W_t, W_{s_1}, \ldots, W_{s_n})$ is a Gaussian random vector, and hence so is $\underline{X} = (W_{t+h} - W_t, W_{s_1}, \ldots, W_{s_n})$ (recall Proposition 3.2.16). The vector $\underline{X}$ has mean $\underline{\mu} = \underline{0}$ and covariance matrix Σ such that $\Sigma_{0k} = \mathbf{E}(W_{t+h} - W_t)W_{s_k} = 0$ for $k = 1, \ldots, n$. In view of Definition 3.2.8 this results with the characteristic function $\Phi_{\underline{X}}(\underline{\theta})$ being the product of the characteristic function of $W_{t+h} - W_t$ and that of $(W_{s_1}, \ldots, W_{s_n})$. Consequently, $W_{t+h} - W_t$ is independent of $(W_{s_1}, \ldots, W_{s_n})$ (see Proposition 3.2.6). Since this applies for any $0 \leq s_1 < s_2 < \ldots < s_n \leq t$, it can be shown that $W_{t+h} - W_t$ is also independent of $\sigma(W_s, s \leq t)$.

In conclusion, the Brownian motion is an example of a zero mean S.P. with independent increments. That is, $(W_{t+h} - W_t)$ is independent of $\{W_s, s \in [0, t]\}$, as stated. ∎

We proceed to construct the Brownian motion as in [**Bre92**, Section 12.7]. To this end, consider

$$L^2([0, T]) = \{f(u) : \int_0^T f^2(u)du < \infty\},$$

equipped with the inner product, $(f, g) = \int_0^T f(u)g(u)du$, where we identify f, g such that $f(t) = g(t)$ for almost every $t \in [0, T]$, as being the same function. As we have seen in Example 2.2.21, this is a separable *Hilbert space*, and there exists a non-random sequence of functions, $\{\phi_i(t)\}_{i=1}^{\infty}$ in $L^2([0, T])$, such that for any

$f, g \in L^2([0,T])$,

$$\lim_{n\to\infty} \sum_{i=1}^{n} (f,\phi_i)(g,\phi_i) = (f,g) \tag{5.1.1}$$

(c.f. Definition 2.2.17 and Theorem 2.2.20). Let X_i be i.i.d., Gaussian R.V.-s with $\mathbf{E}X_i = 0$ and $\mathbf{E}X_i^2 = 1$, all of which are defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For each positive integer N define the stochastic process

$$V_t^N = \sum_{i=1}^{N} X_i \int_0^t \phi_i(u)du\,.$$

Since $\phi_i(t)$ are non-random and any linear combination of the coordinates of a Gaussian random vector gives a Gaussian random vector (see Proposition 3.2.16), we see that V_t^N is a Gaussian process.

We shall show that the random variables V_t^N converge in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ to some random variable V_t, for any fixed, non-random, $t \in [0,T]$. Moreover, we show that the S.P. V_t has properties (a) and (b) of Definition 5.1.1. Then, applying Kolmogorov's continuity theorem, we deduce that the continuous modification of the S.P. V_t is the Brownian motion.

Our next result provides the first part of this program.

Proposition 5.1.3. *Fixing $t \in [0,T]$, the sequence $N \mapsto V_t^N$ is a Cauchy sequence in the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Consequently, there exists a S.P. $V_t(\omega)$ such that $\mathbf{E}[(V_t - V_t^N)^2] \to 0$ as $N \to \infty$, for any $t \in [0,T]$. The S.P. V_t is Gaussian with $\mathbf{E}(V_t) = 0$ and $\mathbf{E}(V_t V_s) = \min(t,s)$.*

Proof. Fix $t \in [0,T]$, noting that for any i,

$$\int_0^t \phi_i(u)du = \int_0^T \mathbf{1}_{[0,t]}(u)\phi_i(u)du = (\mathbf{1}_{[0,t]}, \phi_i)\,. \tag{5.1.2}$$

Set $V_t^0 = 0$ and let

$$\psi_n(t) = \sum_{i=n+1}^{\infty} (\mathbf{1}_{[0,t]}, \phi_i)^2\,.$$

Since $\mathbf{E}(X_i X_j) = \mathbf{1}_{i=j}$ we have for any $N > M \geq 0$,

$$\begin{aligned}
\mathbf{E}\big[(V_t^N - V_t^M)^2\big] &= \sum_{i=M+1}^{N} \sum_{j=M+1}^{N} \mathbf{E}[X_i X_j]\Big(\int_0^t \phi_i(u)du\Big)\Big(\int_0^t \phi_j(u)du\Big) \\
&= \sum_{i=M+1}^{N} \Big(\int_0^t \phi_i(u)du\Big)^2 = \psi_M(t) - \psi_N(t) \qquad (5.1.3)
\end{aligned}$$

(using (5.1.2) for the rightmost equality). Applying (5.1.1) for $f = g = \mathbf{1}_{[0,t]}(\cdot)$ we have that for all M,

$$\psi_M(t) \leq \psi_0(t) = \sum_{i=1}^{\infty} (\mathbf{1}_{[0,t]}, \phi_i)^2 = (\mathbf{1}_{[0,t]}, \mathbf{1}_{[0,t]}) = t < \infty.$$

In particular, taking $M = 0$ in (5.1.3) we see that $\mathbf{E}[(V_t^N)^2]$ are finite for all N. It further follows from the finiteness of the infinite series $\psi_0(t)$ that $\psi_n(t) \to 0$ as $n \to \infty$. In view of (5.1.3) we deduce that V_t^N is a Cauchy sequence in

$L^2(\Omega, \mathcal{F}, \mathbf{P})$, converging to some random variable V_t by the completeness of this space (see Proposition 1.3.20).

Being the pointwise (in t) limit in 2-mean of Gaussian processes, the S.P. V_t is also Gaussian, with the mean and auto-covariance functions for V_t being the (pointwise in t) limits of those for V_t^N (c.f. Proposition 3.2.20). Recall that $\mathbf{E}(V_t^N) = \sum_{i=1}^{N} \mathbf{E}X_i \int_0^t \phi_i(u)du = 0$, for all N, hence $\mathbf{E}(V_t) = 0$ as well.

Repeating the argument used when deriving (5.1.3) we see that for any $s, t \in [0, T]$,

$$\mathbf{E}(V_t^N V_s^N) = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbf{E}[X_i X_j](\int_0^t \phi_i(u)du)(\int_0^s \phi_j(u)du) = \sum_{i=1}^{N} (\mathbf{1}_{[0,t]}, \phi_i)(\mathbf{1}_{[0,s]}, \phi_i)\,.$$

Applying (5.1.1) for $f = \mathbf{1}_{[0,t]}(\cdot)$ and $g = \mathbf{1}_{[0,s]}(\cdot)$, both in $L^2([0, T])$, we now have that

$$\begin{aligned} \mathbf{E}(V_t V_s) &= \lim_{N\to\infty} \mathbf{E}(V_t^N V_s^N) = \lim_{N\to\infty} \sum_{i=1}^{N} (\mathbf{1}_{[0,t]}, \phi_i)(\mathbf{1}_{[0,s]}, \phi_i) \\ &= (\mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}) = \min(t, s)\,, \end{aligned}$$

as needed to conclude the proof of the proposition. ∎

Having constructed a Gaussian stochastic process V_t with the same distribution as a Brownian motion, we next apply *Kolmogorov's continuity theorem*, so as to obtain its continuous modification. This modification is then a Brownian motion. To this end, recall that a *Gaussian* R.V. Y with $\mathbf{E}Y = 0, \mathbf{E}Y^2 = \sigma^2$ has *moments* $\mathbf{E}(Y^{2n}) = \frac{(2n)!}{2^n n!}\sigma^{2n}$. In particular, $\mathbf{E}(Y^4) = 3(\mathbf{E}(Y^2))^2$. Since V_t is Gaussian with

$$\mathbf{E}[(V_{t+h} - V_t)^2] = \mathbf{E}[(V_{t+h} - V_t)V_{t+h}] - \mathbf{E}[(V_{t+h} - V_t)V_t] = h\,,$$

for all t and $h > 0$, we get that

$$\mathbf{E}[(V_{t+h} - V_t)^4] = 3[\mathbf{E}(V_{t+h} - V_t)^2]^2 = 3h^2,$$

as needed to apply Kolmogorov's theorem (with $\alpha = 4$, $\beta = 1$ and $c = 3$ there).

Remark. There is an alternative direct construction of the Brownian motion as the limit of time-space rescaled random walks (see Theorem 3.1.3 for details). Further, though we constructed the Brownian motion W_t as a stochastic process on $[0, T]$ for some finite $T < \infty$, it easily extends to a process on $[0, \infty)$, which we thus take hereafter as the index set of the Brownian motion.

The Brownian motion has many interesting scaling properties, some of which are summarized in your next two exercises.

Exercise 5.1.4. *Suppose W_t is a Brownian motion and $\alpha, s, T > 0$ are non-random constants. Show the following.*

(a) *(Symmetry) $\{-W_t, t \geq 0\}$ is a Brownian motion.*
(b) *(Time homogeneity) $\{W_{s+t} - W_s, t \geq 0\}$ is a Brownian motion.*
(c) *(Time reversal) $\{W_T - W_{T-t}, 0 \leq t \leq T\}$ is a Brownian motion.*
(d) *(Scaling, or self-similarity) $\{\sqrt{\alpha}W_{t/\alpha}, t \geq 0\}$ is a Brownian motion.*
(e) *(Time inversion) If $\tilde{W}_0 = 0$ and $\tilde{W}_t = tW_{1/t}$, then $\{\tilde{W}_t, t \geq 0\}$ is a Brownian motion.*

(f) *With W_t^i denoting independent Brownian motions find the constants c_n such that $c_n \sum_{i=1}^n W_t^i$ are also Brownian motions.*

Exercise 5.1.5. *Fix $\rho \in [-1,1]$. Let $\widetilde{W}_t = \rho W_t^1 + \sqrt{1-\rho^2} W_t^2$ where W_t^1 and W_t^2 are two independent Brownian motions. Show that $\widetilde{W}_t$ is a Brownian motion and find the value of $\mathbf{E}(W_t^1 \widetilde{W}_t)$.*

Exercise 5.1.6. *Fixing $s > 0$ show that the S.P. $\{W_s - W_{s-t}, 0 \le t \le s\}$ and $\{W_{s+t} - W_s, t \ge 0\}$ are two independent Brownian motions and for $0 < t \le s$ evaluate $q_t = \mathbf{P}(W_s > W_{s-t} > W_{s+t})$.*

Applying Doob's inequality you are to prove next the *law of large numbers* for Brownian motion, namely, that almost surely $t^{-1} W_t \to 0$ for $t \to \infty$ (compare with the more familiar law of large numbers, $n^{-1}[S_n - \mathbf{E}S_n] \to 0$ for a random walk S_n).

Exercise 5.1.7. *Let W_t be a Brownian motion.*

(a) *Use the inequality (4.4.6) to show that for any $0 < u < v$,*

$$\mathbf{E}\Big[(\sup_{u \le t \le v} |W_t|/t)^2\Big] \le \frac{4v}{u^2}.$$

(b) *Taking $u = 2^n$ and $v = 2^{n+1}$, $n \ge 1$ in part (a), apply Markov's inequality to deduce that for any $\epsilon > 0$,*

$$\mathbf{P}\Big(\sup_{2^n \le t \le 2^{n+1}} |W_t|/t > \epsilon\Big) \le 8\epsilon^{-2} 2^{-n}.$$

(c) *Applying Borel-Cantelli lemma I conclude that almost surely $t^{-1} W_t \to 0$ for $t \to \infty$.*

Many important S.P. are derived from the Brownian motion W_t. Our next two exercises introduce a few of these processes, the *Brownian bridge* $B_t = W_t - \min(t,1) W_1$, the *Geometric Brownian motion* $Y_t = e^{W_t}$, and the *Ornstein-Uhlenbeck process* $U_t = e^{-t/2} W_{e^t}$. We also define $X_t = x + \mu t + \sigma W_t$, a *Brownian motion with drift* $\mu \in \mathbb{R}$ and diffusion coefficient $\sigma > 0$ starting from $x \in \mathbb{R}$. (See Figure 2 for illustrations of sample paths associated with these processes.)

Exercise 5.1.8. *Compute the mean and the auto-covariance functions of the processes B_t, Y_t, U_t, and X_t. Justify your answers to:*

(a) *Which of the processes W_t, B_t, Y_t, U_t, X_t is Gaussian?*
(b) *Which of these processes is stationary?*
(c) *Which of these processes has continuous sample path?*
(d) *Which of these processes is adapted to the filtration $\sigma(W_s, s \le t)$ and which is also a sub-martingale for this filtration?*

Exercise 5.1.9. *Show that for $0 \le t \le 1$ each of the following S.P. has the same distribution as the Brownian bridge and explain why both have continuous modifications.*

(a) $\widehat{B}_t = (1-t) W_{t/(1-t)}$ *for $t < 1$ with $\widehat{B}_1 = 0$.*
(b) $Z_t = t W_{1/t - 1}$ *for $t > 0$ with $Z_0 = 0$.*

Exercise 5.1.10. *Let $X_t = \int_0^t W_s ds$ for a Brownian motion W_t.*

(a) *Verify that X_t is a well defined stochastic process. That is, check that $\omega \mapsto X_t(\omega)$ is a random variable for each fixed $t \ge 0$.*

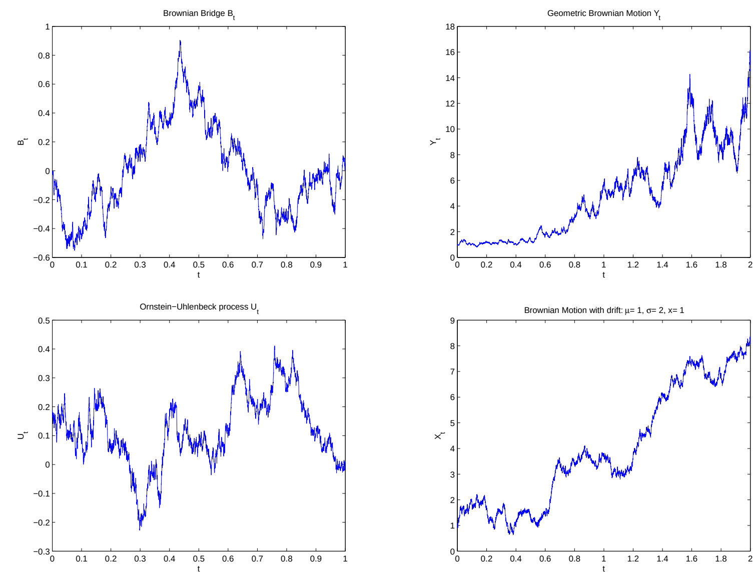

FIGURE 2. Illustration of sample paths for processes in Exercise 5.1.8.

(b) *Using Fubini's theorem 3.3.10 find* $\mathbf{E}(X_t)$ *and* $\mathbf{E}(X_t^2)$.
(c) *Is* X_t *a Gaussian process? Does it have continuous sample paths a.s.? Does it have stationary increments? Independent increments?*

Exercise 5.1.11. *Suppose* W_t *is a Brownian motion.*

(a) *Compute the* probability density function *of the random vector* (W_s, W_t). *Then compute* $\mathbf{E}(W_s|W_t)$ *and* $\text{Var}(W_s|W_t)$, *first for* $s > t$, *then for* $s < t$.
Hint: *Consider Example 2.4.5.*
(b) *Explain why the Brownian Bridge* $\{B_t, 0 \le t \le 1\}$ *has the same distribution as* $\{W_t, 0 \le t \le 1,$ *conditioned upon* $W_1 = 0\}$ *(which is the reason for naming* B_t *a Brownian bridge).*
Hint: *Both Exercise 2.4.6 and parts of Exercise 5.1.8 may help here.*

We conclude with the *fractional Brownian motion*, another Gaussian S.P. of considerable interest in financial mathematics and analysis of computer network traffic.

Exercise 5.1.12. *Fix* $H \in (0, 1)$. *A Gaussian stochastic process* $\{X_t, t \ge 0\}$, *is called a* fractional Brownian motion *(or in short, fBM), of Hurst parameter* H *if* $\mathbf{E}(X_t) = 0$ *and*

$$\mathbf{E}(X_t X_s) = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}] \,, \; s, t \ge 0.$$

(a) *Show that an fBM of Hurst parameter* H *has a continuous modification that is also locally Hölder continuous with exponent* γ *for any* $0 < \gamma < H$.
(b) *Verify that in case* $H = 1/2$ *such modification yields the (standard) Brownian motion.*
(c) *Show the self-similarity property, whereby for any non-random* $\alpha > 0$ *the process* $\{\alpha^H X_{t/\alpha}\}$ *is an fBM of the same Hurst parameter* H.

(d) *For which values of H is the fBM a process of stationary increments and for which values of H is it a process of independent increments?*

5.2. The reflection principle and Brownian hitting times

We start with Paul Lévy's martingale characterization of the Brownian motion, stated next.

Theorem 5.2.1 (Lévy's martingale characterization)**.** *Suppose square-integrable MG $(X_t, \mathcal{F}_t)$ of right-continuous filtration and continuous sample path is such that $(X_t^2 - t, \mathcal{F}_t)$ is also a MG. Then, X_t is a Brownian motion.*

Remark. The continuity of X_t is *essential* for Lévy's martingale characterization. For example, the square-integrable martingale $X_t = N_t - t$, with N_t the Poisson process of rate one (per Definition 6.2.1), is such that $X_t^2 - t$ is also a MG (see Exercise 6.2.2). Of course, almost all sample path of the Poisson process are discontinuous.

A consequence of this characterization is that a square-integrable MG with continuous sample path and unbounded increasing part is merely a *time changed Brownian motion* (c.f. [**KS97**, Theorem 3.4.6]).

Proposition 5.2.2. *Suppose $(X_t, \mathcal{F}_t)$ is a square-integrable martingale with $X_0 = 0$, right-continuous filtration and continuous sample path. If the increasing part A_t in the corresponding Doob-Meyer decomposition of Theorem 4.4.7 is almost surely unbounded then $W_s = X_{\tau_s}$ is a Brownian motion, where $\tau_s = \inf\{t \geq 0 : A_t > s\}$ are $\mathcal{F}_t$-stopping times such that $s \mapsto \tau_s$ is non-decreasing and right-continuous mapping of $[0, \infty)$ to $[0, \infty)$, with $A_{\tau_s} = s$ and $X_t = W_{A_t}$.*

Our next proposition may be viewed as yet another application of Lévy's martingale characterization. In essence it states that each stopping time acts as a *regeneration point* for the Brownian motion. In particular, it implies that the Brownian motion is a strong Markov process (in the sense of Definition 6.1.21). As we soon see, this "regeneration" property is very handy for finding the distribution of certain Brownian hitting times and running maxima.

Proposition 5.2.3. *If τ is a stopping time for the canonical filtration $\mathcal{G}_t$ of the Brownian motion W_t then the S.P. $X_t = W_{t+\tau} - W_\tau$ is also a Brownian motion, which is independent of the stopped σ-field $\mathcal{G}_\tau$.*

Remark. This result is stated as [**Bre92**, Theorem 12.42], with a proof that starts with a stopping time τ taking a countable set of values and moves to the general case by approximation, using sample path continuity. Alternatively, with the help of some amount of stochastic calculus one may verify the conditions of Lévy's theorem for X_t and the filtration $\mathcal{F}_t = \sigma(W_{s+\tau} - W_\tau, 0 \leq s \leq t)$. We will detail neither approach here.

We next apply Proposition 5.2.3 for computing the probability density function of the *first hitting time* $\tau_\alpha = \inf\{t > 0 : W_t = \alpha\}$ for any fixed $\alpha > 0$. Since the Brownian motion has continuous sample path, we know that $\tau_\alpha = \min\{t > 0 : W_t = \alpha\}$ and that the maximal value of W_t for $t \in [0, T]$ is always achieved at some $t \leq T$. Further, since $W_0 = 0 < \alpha$, if $W_s \geq \alpha$ for some $s > 0$, then $W_u = \alpha$ for some $u \in [0, s]$, that is, $\tau_\alpha \leq s$ with $W_{\tau_\alpha} = \alpha$. Consequently,

$$\{\omega : W_T(\omega) \geq \alpha\} \subseteq \{\omega : \max_{0 \leq s \leq T} W_s(\omega) \geq \alpha\} = \{\omega : \tau_\alpha(\omega) \leq T\} .$$

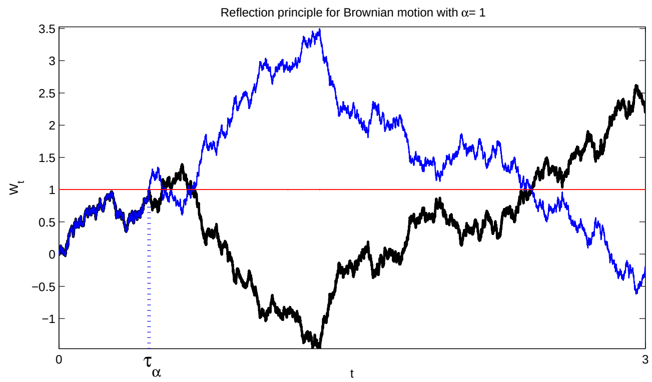

FIGURE 3. Illustration of the reflection principle for Brownian motion.

Recall that $X_t = W_{t+\tau_\alpha} - W_{\tau_\alpha}$ is a Brownian motion, independent of the random variable τ_α (which is measurable on $\mathcal{G}_{\tau_\alpha}$). In particular, the law of X_t is invariant to a sign-change, so we have the *reflection principle* for the Brownian motion, stating that

$$\begin{aligned} \mathbf{P}(\max_{0\le s\le T} W_s \ge \alpha,\ W_T \ge \alpha) &= \mathbf{P}(\tau_\alpha \le T, X_{T-\tau_\alpha} \ge 0) \\ = \mathbf{P}(\tau_\alpha \le T, X_{T-\tau_\alpha} \le 0) &= \mathbf{P}(\max_{0\le s\le T} W_s \ge \alpha,\ W_T \le \alpha)\,. \end{aligned}$$

Also, $\mathbf{P}(W_T = \alpha) = 0$ and we have that

$$\begin{aligned} \mathbf{P}(\max_{0\le s\le T} W_s \ge \alpha) &= \mathbf{P}(\max_{0\le s\le T} W_s \ge \alpha, W_T \ge \alpha) + \mathbf{P}(\max_{0\le s\le T} W_s \ge \alpha, W_T \le \alpha) \\ &= 2\mathbf{P}(\max_{0\le s\le T} W_s \ge \alpha, W_T \ge \alpha) = 2\mathbf{P}(W_T \ge \alpha) \qquad (5.2.1) \\ &= 2\int_{\alpha T^{-1/2}}^{\infty} e^{\frac{-x^2}{2}} \frac{dx}{\sqrt{2\pi}} \end{aligned}$$

Among other things, this shows that $\mathbf{P}(\tau_\alpha > T) \to 0$ as $T \to \infty$, hence $\tau_\alpha < \infty$ with probability one. Further, we have that the probability density function of τ_α at T is given by

$$p_{\tau_\alpha}(T) = \frac{\partial[\mathbf{P}(\tau_\alpha \le T)]}{\partial T} = 2\frac{\partial}{\partial T}\int_{\alpha T^{-1/2}}^{\infty} e^{\frac{-x^2}{2}} \frac{dx}{\sqrt{2\pi}} = \frac{\alpha}{\sqrt{2\pi}T^{3/2}} e^{-\frac{\alpha^2}{2T}}\,. \qquad (5.2.2)$$

This computation demonstrates the power of the reflection principle and more generally, that many computations for stochastic processes are the most explicit when they are done for the Brownian motion.

Our next exercise provides yet another example of a similar nature.

Exercise 5.2.4. *Let W_t be a Brownian motion.*

(a) *Show that $-\min_{0\le t\le T} W_t$ and $\max_{0\le t\le T} W_t$ have the same distribution which is also the distribution of $|W_T|$.*

(b) *Show that the probability α that the Brownian motion W_u attains the value zero at some $u \in (s, s+t)$ is given by $\alpha = \int_{-\infty}^{\infty} p_t(|x|)\phi_s(x)dx$, where $p_t(x) = \mathbf{P}(|W_t| \ge x)$ for $x, t > 0$ and $\phi_s(x)$ denotes the probability density of the R.V. W_s.*

Remark: *The explicit formula* $\alpha = (2/\pi)\arccos(\sqrt{s/(s+t)})$ *is obtained in* [**KT75**, page 348] *by computing this integral.*

Remark. Using a *reflection principle* type argument one gets the discrete time analog of (5.2.1), whereby the simple random walk S_n of Definition 3.1.2 satisfies for each integer $r > 0$ the identity

$$\mathbf{P}(\max_{0\le k\le n} S_k \ge r) = 2\mathbf{P}(S_n > r) + \mathbf{P}(S_n = r)\,.$$

Fixing $\alpha > 0$ and $\beta > 0$ consider the stopping time $\tau_{\beta,\alpha} = \inf\{t : W_t \ge \alpha$ or $W_t \le -\beta\}$ (for the canonical filtration of the Brownian motion W_t). By continuity of the Brownian sample path we know that $W_{\tau_{\beta,\alpha}} \in \{\alpha, -\beta\}$. Applying Doob's optional stopping theorem for the uniformly integrable stopped martingale $W_{t\wedge\tau_{\beta,\alpha}}$ of continuous sample path we get that $\mathbf{P}(W_{\tau_{\beta,\alpha}} = \alpha) = \beta/(\alpha+\beta)$ (for more details see Exercise 4.3.18).

Exercise 5.2.5. *Show that* $\mathbf{E}(\tau_{\beta,\alpha}) = \alpha\beta$ *by applying Doob's optional stopping theorem for the uniformly integrable stopped martingale* $W^2_{t\wedge\tau_{\beta,\alpha}} - t\wedge\tau_{\beta,\alpha}$.

We see that the expected time it takes the Brownian motion to exit the interval $(-\beta, \alpha)$ is finite for any finite α and β. As $\beta \uparrow \infty$, these exit times $\tau_{\beta,\alpha}$ converge monotonically to the time of reaching level α, namely $\tau_\alpha = \inf\{t > 0 : W_t = \alpha\}$. Exercise 5.2.5 implies that τ_α has infinite expected value (we can see this also directly from the formula (5.2.2) for its probability density function).

To summarize, the Brownian motion eventually reach any level, the expected time it takes for doing so is infinite, while the exit time of any finite interval has finite mean (and moreover, all its moments are finite).

Building on Exercises 4.2.9 and 5.2.5 here is an interesting fact about the *planar Brownian motion.*

Exercise 5.2.6. *The* planar Brownian motion *is an* $\mathbb{R}^2$*-valued stochastic process* $\underline{W}_t = (X_t, Y_t)$ *consisting of two independent Brownian motions* $\{X_t\}$ *and* $\{Y_t\}$. *Let* $R_t = \sqrt{X_t^2 + Y_t^2}$ *denote its distance from the origin and* $\theta_r = \inf\{t : R_t \ge r\}$ *the corresponding first hitting time for a sphere of radius* $r > 0$ *around the origin.*

(a) *Show that* $M_t = R_t^2 - 2t$ *is a martingale for* $\mathcal{F}_t = \sigma(X_s, Y_s, s \le t)$.
Hint: *Consider Proposition 2.3.17.*
(b) *Check that* $\theta_r \le \tau_{r,r} = \inf\{t : |X_t| \ge r\}$ *and that* θ_r *is a stopping time for the filtration* $\mathcal{F}_t$.
(c) *Verify that* $\{M_{t\wedge\theta_r}\}$ *is uniformly integrable and deduce from Doob's optional stopping theorem that* $\mathbf{E}[\theta_r] = r^2/2$.

5.3. Smoothness and variation of the Brownian sample path

We start with a definition of the q-th variation of a function $f(t)$ on a finite interval $t \in [a, b]$, $a < b$ of the real line, where $q \ge 1$. We shall study here only the *total variation*, corresponding to $q = 1$ and the *quadratic variation*, corresponding to $q = 2$.

Definition 5.3.1. *For any finite partition* π *of* $[a, b]$, *that is,* $\pi = \{a = t_0^{(\pi)} < t_1^{(\pi)} < \ldots < t_k^{(\pi)} = b\}$, *let* $\|\pi\| = \max_i\{t_{i+1}^{(\pi)} - t_i^{(\pi)}\}$ *denote the length of the longest* *interval in* π *and*

$$V^{(q)}_{(\pi)}(f) = \sum_i |f(t^{(\pi)}_{i+1}) - f(t^{(\pi)}_i)|^q$$

denote the q*-th variation of* $f(\cdot)$ *on* π*. The* q-th variation *of* $f(\cdot)$ *on* $[a, b]$ *is then*

$$V^{(q)}(f) = \lim_{\|\pi\|\to 0} V^{(q)}_{(\pi)}(f) \,, \tag{5.3.1}$$

provided such limit exists.

We next extend this definition to continuous time stochastic processes.

Definition 5.3.2. *The* q-th variation *of a S.P.* X_t *on the interval* $[a, b]$ *is the random variable* $V^{(q)}(X)$ *obtained when replacing* $f(t)$ *by* $X_t(\omega)$ *in the above definition, provided the limit (5.3.1) exists (in some sense).*

The quadratic variation is affected by the smoothness of the sample path. For example, suppose that a S.P. $X(t)$ has Lipschitz sample path with probability one. Namely, there exists a random variable $L(\omega)$ which is finite almost surely, such that $|X(t) - X(s)| \le L|t - s|$ for all $t, s \in [a, b]$. Then,

$$\begin{aligned} V^{(2)}_{(\pi)}(X) &\le L^2 \sum_i (t^{(\pi)}_{i+1} - t^{(\pi)}_i)^2 \\ &\le L^2\|\pi\| \sum_i (t^{(\pi)}_{i+1} - t^{(\pi)}_i) = L^2\|\pi\|(b - a)\,, \end{aligned} \tag{5.3.2}$$

converges to zero almost surely as $\|\pi\| \to 0$. So, such a S.P. has zero quadratic variation on $[a, b]$.

By considering different time intervals we view the quadratic variation as yet another stochastic process.

Definition 5.3.3. *The* quadratic variation *of a stochastic process* X*, denoted* $V^{(2)}_t(X)$ *is the non-decreasing, non-negative S.P. corresponding to the quadratic variation of* X *on the intervals* $[0, t]$.

Focusing hereafter on the Brownian motion, we have that,

Proposition 5.3.4. *For a Brownian motion* $W(t)$*, as* $\|\pi\| \to 0$ *we have that* $V^{(2)}_{(\pi)}(W) \to (b - a)$ *in* 2*-mean.*

Proof. Fixing a finite partition π, note that

$$\begin{aligned} \mathbf{E}[V^{(2)}_{(\pi)}(W)] &= \sum_i \mathbf{E}[(W(t_{i+1}) - W(t_i))^2] \\ &= \sum_i \mathrm{Var}(W(t_{i+1}) - W(t_i)) = \sum_i (t_{i+1} - t_i) = b - a \,. \end{aligned}$$

Similarly, by the independence of increments,

$$\begin{aligned} \mathbf{E}[V^{(2)}_{(\pi)}(W)^2] &= \sum_{i,j} \mathbf{E}[(W(t_{i+1}) - W(t_i))^2\,(W(t_{j+1}) - W(t_j))^2] \\ &= \sum_i \mathbf{E}[(W(t_{i+1}) - W(t_i))^4] \\ &\quad + \sum_{i\neq j} \mathbf{E}[(W(t_{i+1}) - W(t_i))^2]\,\mathbf{E}[(W(t_{j+1}) - W(t_j))^2] \end{aligned}$$

Since $W(t_{j+1}) - W(t_j)$ is Gaussian of mean zero and variance $(t_{j+1} - t_j)$, it follows that

$$\mathbf{E}[V_{(\pi)}^{(2)}(W)^2] = 3\sum_i (t_{i+1}-t_i)^2 + \sum_{i\neq j}(t_{i+1}-t_i)(t_{j+1}-t_j) = 2\sum_i (t_{i+1}-t_i)^2 + (b-a)^2 .$$

So, $\mathrm{Var}(V_{(\pi)}^{(2)}(W)) = \mathbf{E}(V_{(\pi)}^{(2)}(W)^2) - (b-a)^2 \leq 2\|\pi\|(b-a) \to 0$ as $\|\pi\| \to 0$. With the mean of $V_{(\pi)}^{(2)}(W)$ being $(b-a)$ and its variance converging to zero, we have the stated convergence in 2-mean. ∎

Here are two consequences of Proposition 5.3.4.

Corollary 5.3.5. *The quadratic variation of the Brownian motion is the S.P. $V_t^{(2)}(W) = t$, which is the same as the* increasing process *in the Doob-Meyer decomposition of W_t^2. More generally, the quadratic variation equals the increasing process for any* square-integrable martingale *of continuous sample path and right-continuous filtration (as shown for example in* [**KS97**, Theorem 1.5.8, page 32]*).*

Remark. Since $V_{(\pi)}^{(2)}$ are observable on the sample path, considering finer and finer partitions π_n, one may numerically estimate the quadratic variation for a given sample path of a S.P. The quadratic variation of the Brownian motion is non-random, so if this numerical estimate significantly deviates from t, we conclude that Brownian motion is not a good model for the given S.P.

Corollary 5.3.6. *With probability one, the sample path of the Brownian motion $W(t)$ is* not Lipschitz continuous *in any interval $[a, b]$, $a < b$.*

Proof. Fix a finite interval $[a, b]$, $a < b$ and let Γ_L denote the set of outcomes ω for which $|W(t) - W(s)| \leq L|t - s|$ for all $t, s \in [a, b]$. From (5.3.2) we see that if $\|\pi\| \leq 1/(2L^2)$ then

$$\mathrm{Var}(V_{(\pi)}^{(2)}(W)) \geq \mathbf{E}[(V_{(\pi)}^{(2)}(W) - (b-a))^2 I_{\Gamma_L}] \geq \frac{(b-a)^2}{4}\mathbf{P}(\Gamma_L) .$$

By Proposition 5.3.4 we know that $\mathrm{Var}(V_{(\pi)}^{(2)}(W)) \to 0$ as $\|\pi\| \to 0$, hence necessarily $\mathbf{P}(\Gamma_L) = 0$. As the set Γ of outcomes for which the sample path of $W(t)$ is Lipschitz continuous is just the (countable) union of Γ_L over positive integer values of L, it follows that $\mathbf{P}(\Gamma) = 0$, as stated. ∎

We can even improve upon this negative result as following.

Exercise 5.3.7. *Fixing $\gamma > \frac{1}{2}$ check that by the same type of argument as above, with probability one, the sample path of the Brownian motion is* not globally Hölder continuous *of exponent γ in any interval $[a, b]$, $a < b$.*
In contrast, applying Theorem 3.3.3 verify that with probability one the sample path of the Brownian motion is locally Hölder continuous for any exponent $\gamma < 1/2$ (see part (c) of Exercise 3.3.5 for a similar derivation).

The next exercise shows that one can strengthen the convergence of the quadratic variation for $W(t)$ by imposing some restrictions on the allowed partitions.

Exercise 5.3.8. *Let $V_{(\pi)}^{(2)}(W)$ denote the approximation of the quadratic variation of the Brownian motion for a finite partition π of $[a, a + t]$. Combining Markov's* *inequality (for $f(x) = x^2$) and Borel-Cantelli I show that for the Brownian motion $V^{(2)}_{(\pi_n)}(W) \xrightarrow{a.s.} t$ if the finite partitions π_n are such that $\sum_n \|\pi_n\| < \infty$.*

In the next exercise, you are to follow a similar procedure, en-route to finding the quadratic variation for a Brownian motion with drift.

Exercise 5.3.9. *Let $Z(t) = W(t) + rt$, $t \geq 0$, where $W(t)$ is a Brownian motion and r a non-random constant.*

(a) *What is the law of $Y = Z(t+h) - Z(t)$?*
(b) *For which values of $t' < t$ and $h, h' > 0$ are the variables Y and $Y' = Z(t'+h') - Z(t')$ independent?*
(c) *Find the quadratic variation $V_t^{(2)}(Z)$ of the stochastic process $\{Z(t)\}$.* Hint: *See Exercise 5.3.15.*

Typically, the stochastic integral $I_t = \int_0^t X_s dW_s$ is first constructed in case X_t is a "simple" process (that is having sample path that are piecewise constant on non-random intervals), exactly as you do next.

Exercise 5.3.10. *Suppose $(W_t, \mathcal{F}_t)$ satisfies Lévy's characterization of the Brownian motion. Namely, it is a square-integrable martingale of right-continuous filtration and continuous sample path such that $(W_t^2 - t, \mathcal{F}_t)$ is also a martingale. Suppose X_t is a bounded $\mathcal{F}_t$-adapted* simple process. *That is,*

$$X_t = \eta_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} \eta_i \mathbf{1}_{(t_i, t_{i+1}]}(t) ,$$

where the non-random sequence $t_k > t_0 = 0$ is strictly increasing and unbounded (in k), while the (discrete time) S.P. $\{\eta_n\}$ is uniformly (in n and ω) bounded and adapted to $\mathcal{F}_{t_n}$. Provide an explicit formula for $A_t = \int_0^t X_u^2 du$, then show that both

$$I_t = \sum_{j=0}^{k-1} \eta_j (W_{t_{j+1}} - W_{t_j}) + \eta_k (W_t - W_{t_k}), \ \text{when } t \in [t_k, t_{k+1}) ,$$

and $I_t^2 - A_t$ are martingales with respect to $\mathcal{F}_t$ and explain why this implies that $\mathbf{E} I_t^2 = \mathbf{E} A_t$ and $V_t^{(2)}(I) = A_t$.

We move from the quadratic variation $V^{(2)}$ to the *total variation* $V^{(1)}$. Note that, when $q = 1$, the limit in (5.3.1) always exists and equals the supremum over all finite partitions π.

Example 5.3.11. *The total variation is particularly simple for* monotone functions. *Indeed, it is easy to check that if $f(t)$ is monotone then its total variation is $V^{(1)}(f) = \max_{t\in[a,b]}\{f(t)\} - \min_{t\in[a,b]}\{f(t)\}$. In particular, the total variation of monotone functions is finite on finite intervals even though the functions may well be discontinuous.*

In contrast we have that

Proposition 5.3.12. *The total variation of the Brownian motion $W(t)$ is infinite with probability one.*

PROOF. Let $\alpha(h) = \sup_{a \le t \le b-h} |W(t+h) - W(t)|$. With probability one, the sample path $W(t)$ is continuous hence uniformly continuous on the closed, bounded interval $[a,b]$. Therefore, $\alpha(h) \xrightarrow{a.s.} 0$ as $h \to 0$. Let π_n divide $[a,b]$ to 2^n equal parts, so $\|\pi_n\| = 2^{-n}(b-a)$. Then,

$$\begin{aligned} V^{(2)}_{(\pi_n)}(W) &= \sum_{i=0}^{2^n-1} [W(a+(i+1)\|\pi_n\|) - W(a+i\|\pi_n\|)]^2 \\ &\le \alpha(\|\pi_n\|) \sum_{i=0}^{2^n-1} |W(a+(i+1)\|\pi_n\|) - W(a+i\|\pi_n\|)| \,. \end{aligned} \tag{5.3.3}$$

Recall Exercise 5.3.8, that almost surely $V^{(2)}_{(\pi_n)}(W) \to (b-a) < \infty$. This, together with (5.3.3) and the fact that $\alpha(\|\pi_n\|) \xrightarrow{a.s.} 0$, imply that

$$V^{(1)}_{(\pi_n)}(W) = \sum_{i=0}^{2^n-1} |W(a+(i+1)\|\pi_n\|) - W(a+i\|\pi_n\|)| \xrightarrow{a.s.} \infty,$$

implying that $V^{(1)}(W) = \infty$ with probability one, as stated. ∎

Remark. Comparing Example 5.3.11 and Proposition 5.3.12 we have that the sample path of the Brownian motion is almost surely non-monotone on each non-empty open interval. Here is an alternative, direct proof of this result (c.f. [**KS97**, Theorem 2.9.9]).

Exercise 5.3.13. *Let W_t be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.*

(a) *Let $A_n = \bigcap_{i=1}^n \{\omega \in \Omega : W_{i/n}(\omega) - W_{(i-1)/n}(\omega) \ge 0\}$ and $A = \{\omega \in \Omega : t \mapsto W_t(\omega)$ is non-decreasing on $[0,1]\}$. Explain why $A = \cap_n A_n$ why $\mathbf{P}(A_n) = 2^{-n}$ and why it implies that $A \in \mathcal{F}$ and $\mathbf{P}(A) = 0$.*

(b) *Use the symmetry of the Brownian motion's sample path (per Exercise 5.1.4) to deduce that the probability that it is monotone on $[0,1]$ is 0. Verify that the same applies for any interval $[s,t]$ with $0 \le s < t$ non-random.*

(c) *Show that, for almost every ω, the sample path $t \mapsto W_t(\omega)$ is non-monotone on any non-empty open interval.*
Hint: *Let F denote the set of ω such that $t \mapsto W_t(\omega)$ is monotone on some non-empty open interval, observing that*

$$F = \bigcup_{s,t \in \mathbb{Q}, 0 \le s < t} \{\omega \in \Omega : t \mapsto W_t(\omega) \text{ is monotone on } [s,t]\}.$$

To practice your understanding, solve the following exercises.

Exercise 5.3.14. *Consider the stochastic process $Y(t) = W(t)^2$, for $0 \le t \le 1$, with $W(t)$ a Brownian motion.*

(a) *Show that for any $\gamma < 1/2$ the sample path of $Y(t)$ is locally Hölder continuous of exponent γ with probability one.*

(b) *Compute $\mathbf{E}[V^{(2)}_{(\pi)}(Y)]$ for a finite partition π of $[0,t]$ to k intervals, and find its limit as $\|\pi\| \to 0$.*

(c) *Show that the total variation of $Y(t)$ on the interval $[0,1]$ is infinite.*

Exercise 5.3.15.

(a) *Show that if functions $f(t)$ and $g(t)$ on $[a, b]$ have zero and finite quadratic variations, respectively (i.e. $V^{(2)}(f) = 0$ and $V^{(2)}(g) < \infty$ exists), then $V^{(2)}(g + f) = V^{(2)}(g)$.*
(b) *Show that if a (uniformly) continuous function $f(t)$ has finite total variation then $V^{(q)}(f) = 0$ for any $q > 1$.*
(c) *Suppose both X_t and $\widetilde{A}_t$ have continuous sample path, such that $t \mapsto \widetilde{A}_t$ has finite total variation on any bounded interval and X_t is a square-integrable martingale. Deduce that then $V_t^{(2)}(X + \widetilde{A}) = V_t^{(2)}(X)$.*

- *What follows should be omitted at first reading.*

We saw that the sample path of the Brownian motion is rather irregular, for it is neither monotone nor Lipschitz continuous at any open interval. [**Bre92**, Theorem 12.25] somewhat refines the latter conclusion, showing that with probability one the sample path is nowhere differentiable.

We saw that almost surely the sample path of the Brownian motion is Hölder continuous of any exponent $\gamma < \frac{1}{2}$ (see Exercise 5.3.7), and of no exponent $\gamma > \frac{1}{2}$. The exact *modulus of continuity* of the Brownian path is provided by P. Lévy's (1937) theorem (see [**KS97**, Theorem 2.9.25, page 114]):

$$\mathbf{P}(\limsup_{\delta \downarrow 0} \frac{1}{g(\delta)} \sup_{\substack{0 \le s,t \le 1 \\ |t-s| \le \delta}} |W(t) - W(s)| = 1) = 1,$$

where $g(\delta) = \sqrt{2\delta \log(\frac{1}{\delta})}$ for any $\delta > 0$. This means that $|W(t) - W(s)| \le Cg(\delta)$ for any $C > 1$, $\delta > 0$ small enough (possibly depending on ω), and $|t - s| < \delta$.

Many other "irregularity" properties of the Brownian sample path are known. For example ([**KS97**, Theorem 2.9.12]), for almost every ω, the set of points of local maximum for the path is countable and dense in $[0, \infty)$, and all *local maxima* are strict (recall that t is a point of local maximum of $f(\cdot)$ if $f(s) \le f(t)$ for all s in some open interval around t, and it is strict if in this interval also $f(s) < f(t)$ except at $s = t$). Moreover, almost surely, the *zero set* of points t where $W(t) = 0$, is closed, unbounded, of zero Lebesgue measure, with accumulation point at zero and no isolated points (this is [**KS97**, Theorem 2.9.6], or [**Bre92**, Theorem 12.35]). These properties further demonstrate just how wildly the Brownian path change its direction. Try to visualize a path having such properties!

We know that W_t is a Gaussian R.V. of variance t. As such it has the law of $\sqrt{t}W_1$, suggesting that the Brownian path grows like $\sqrt{t}$ as $t \to \infty$. While this is true when considering fixed, non-random times, it ignores the random fluctuations of the path. Accounting for these we obtain the following *Law of the Iterated Logarithm*,

$$\limsup_{t \to \infty} \frac{W_t(\omega)}{\sqrt{2t \log(\log t)}} = 1, \qquad \text{almost surely.}$$

Since $-W_t$ is also a Brownian motion, this is equivalent to

$$\liminf_{t \to \infty} \frac{W_t(\omega)}{\sqrt{2t \log(\log t)}} = -1, \qquad \text{almost surely.}$$

Recall that $tW_{1/t}$ is also a Brownian motion (see Exercise 5.1.4), so the law of the iterated logarithm is equivalent to

$$\limsup_{t\to 0} \frac{W_t(\omega)}{\sqrt{2t\log(\log(\frac{1}{t}))}} = 1 \quad \& \quad \liminf_{t\to 0} \frac{W_t(\omega)}{\sqrt{2t\log(\log(\frac{1}{t}))}} = -1, \qquad \text{almost surely,}$$

providing information on the behavior of W_t for *small* t (for proof, see [**Bre92**, Theorem 12.29]). An immediate consequence of the law of the iterated logarithm is the law of large numbers for Brownian motion (which you have already proved in Exercise 5.1.7).